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Asymptotic Properties of Rapidly Varying Functions*

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ABSTRACT. In this paper we study asymptotic properties of rapidly varying functions. The important properties of this class that are related to arithmetic mean will be proved. The asymptotic properties of series $\sum_{n=1}^{\infty} f(n)$ when f rapidly varying functions will be proved, also.

1. INTRODUCTION

A measurable function $f : [a, \infty) \mapsto (0, \infty) (a > 0)$ is called **regularly varying** in the sense of Karamata if for some $\alpha \in R$ it satisfies

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\alpha}$$

for every $\lambda > 0$, and we denote $f \in R_{\alpha}$.

Class R_{α} were introduced by J. Karamata [3] in 1930. Karamata proved [see e.g. 1,3] that if function $f \in R_{\alpha}, \alpha > 0$, is locally bounded than

$$\int_a^x f(t)dt \sim \frac{x}{\alpha+1} f(x), \qquad (x \to \infty).$$

A measurable function $f : [a, \infty) \mapsto (0, \infty) (a > 0)$ is called **rapidly varying** in the sense of de Haan with the index of variability ∞ if it satisfies

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \infty$$

for every $\lambda > 1$. This functional class is denoted by R_{∞} (see e.g. [1]).

Example. If $f(x) = x^{r(x)}, r(x) \to \infty$, and r is nondecreasing function then $f \in R_{\infty}$.

We using following properties of rapidly varying functions:

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(1) In [1] is proved that if function $f \in R_\infty$ than

$$\lim_{x\to\infty} \frac{\underline{f}(\lambda x)}{f(x)} = \infty$$

and $\underline{f} \in R_\infty$.

(2) Using (1), we have, for $f \in R_\infty$

$$\lim_{x\to\infty} \frac{f(\varphi(x))}{f(\psi(x))} = \infty$$

if $\psi(x) \to \infty$ and $\liminf_{x\to\infty} \frac{\varphi(x)}{\psi(x)} > 1$.

2. Results

Lemma 1. *Let $a > 0$ and $f \in R_\infty$ be a locally bounded function on $[0, \infty)$ then for every $n \in N$ and $\lambda > 1$ exist $y_0 > \lambda a$ that*

$$\frac{f(y)}{f(x)} > \left(\frac{y}{x}\right)^n, \quad \textit{for } y > y_0,\ y > \lambda x,\ x > a.$$

Proof. From $\lim\limits_{t\to\infty} \frac{\underline{f}(\lambda t)}{\underline{f}(t)} = \infty$ exist $t_0 > 0$ with properties: If $t \geq t_0$ than $\frac{\underline{f}(\lambda t)}{\underline{f}(t)} > \lambda^{2n}$. Let $y_0 > t_0$ with properties: if $y \geq y_0$ than $f(y) > \sup\{f(s)|a \leq s \leq t_0\} \left(\frac{y}{a}\right)^n = Cy^n$. Than y_0 exist because f is rapid varying function. Now we consider two cases:

(1) $t_0 \leq x$. Let $k \in N$ have properties $\lambda^k x \leq y < \lambda^{k+1} x$ than

$$\frac{f(y)}{f(x)} > \frac{\underline{f}(\lambda^k x)}{f(x)} > \frac{\underline{f}(\lambda^k x)}{\underline{f}(\lambda^{k-1} x)} \cdots \frac{\underline{f}(\lambda x)}{f(x)} > (\lambda^{2n})^k \geq (\lambda^{k+1})^n > \left(\frac{y}{x}\right)^n.$$

(2) $a \leq x < t_0$, than we have

$$\frac{f(y)}{f(x)} \geq \frac{f(y)}{\sup\{f(t)|a \leq t \leq t_0\}} > \left(\frac{y}{a}\right)^n > \left(\frac{y}{x}\right)^n, y \geq y_0. \qquad \square$$

Lemma 2. *If $x_1, x_2, \ldots, x_n$ are positive real numbers, $\lambda = \frac{x_1+\cdots+x_n}{\mathrm{Max}\,\{x_1,\ldots,x_n\}}$ and $p \geq 1$. Then*

$$(x_1 + \cdots + x_n)^p \geq \lambda^{p-1}(x_1^p + \cdots + x_n^p).$$

Proof.

$$(x_1 + \cdots + x_n)^p = \left(\frac{x_1 + x_2 + \cdots + x_n}{x_1}\right)^{p-1} x_1^p + \left(\frac{x_1 + x_2 + \cdots + x_n}{x_2}\right)^{p-1} x_2^p + \cdots$$
$$+ \left(\frac{x_1 + x_2 \cdots + x_n}{x_n}\right)^{p-1} x_n^p \geq \lambda^{p-1}(x_1^p + \cdots + x_n^p) \qquad \square$$

Theorem 1. *Let $f \in R_\infty$ be a locally bounded function on $[0,\infty)$ and $(x_n)_{n=1}^\infty$ sequence with properties $\inf\{x_n | n \in N\} > 0$ and $\liminf\limits_{n\to\infty} \frac{x_1+\cdots+x_n}{\text{Max}\,\{x_1,\ldots,x_n\}} > 1$. Then*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ll f(x_1 + x_2 + \cdots + x_n), \qquad (n \to \infty).$$

Proof. Let $M > 1$ and $a = \inf_{n\in N} x_n > 0$ and $\lambda = \liminf\limits_{n\to\infty} \frac{x_1+\cdots+x_n}{\text{Max}\,\{x_1,\ldots,x_n\}} > 1$, obviously $\lim\limits_{n\to\infty}(x_1 + \cdots + x_n) = \infty$.

Using Lemma 1, for sufficiently large n, $k \in \{1, 2, \ldots, n\}$ and every $p > 1$ we have

$$\frac{f(x_1 + x_2 + \cdots + x_n)}{f(x_k)} > \left(\frac{x_1 + x_2 + \cdots + x_n}{x_k}\right)^p.$$

Using Lemma 2

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{f(x_1 + x_2 + \cdots + x_n)} \le \frac{x_1^p + \cdots + x_n^p}{(x_1 + \cdots + x_n)^p} \le \frac{1}{\lambda^{p-1}}.$$

If we put $p = \frac{lnM}{ln\lambda} + 1$ we have

$$f(x_1 + \cdots + x_n) > M(f(x_1 + \cdots + f(x_n)).$$ □

Corollary 1. *Let it (x_n) positive and nondecreasing sequence. For every $f \in R_\infty$.*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ll f(x_1 + x_2 + \cdots + x_n), \qquad (n \to \infty)$$

if and only if

$$\liminf_{n\to\infty} \frac{x_1 + \cdots + x_n}{x_n} > 1.$$

Proof. Proof of direct way we have from Theorem 1.

If $\liminf\limits_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{x_n} = 1$ we have index $(k_n)_{n=1}^\infty$ with properties

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_{k_n}}{x_{k_n}} = 1.$$

Now we defined function f

$$f(x) = e^x \quad \text{for } x \in (0, +\infty),\ x \neq x_{k_i},\ i \in N$$
$$f(x_{k_i}) = e^{x_1+x_2+\cdots+x_{k_i}}, \quad i \in N.$$

First we will show $f \in R_\infty$.

For $\lambda > 1$ holds $\lambda - \frac{x_1+\cdots+x_{k_n}}{x_{k_n}} > \frac{\lambda-1}{2} > 0$ for sufficiently large n. Then we have

$$\frac{e^{\lambda x_{k_n}}}{e^{x_1+x_2+\cdots+x_{k_n}}} = \left(e^{\lambda - \frac{x_1+\cdots+x_{k_n}}{x_{k_n}}}\right)^{x_{k_n}} \to \infty, \qquad (n \to \infty).$$

For $x_{k_n} \leq x < x_{k_{n+1}}$ we have

$$\frac{f(\lambda x)}{f(x)} > \frac{e^{\lambda x}}{\max(e^{x_1+\cdots+x_{n_k}}, e^x)} \to \infty, \qquad (x \to \infty).$$

From this observation we have $f \in R_\infty$, but

$$e^{x_1+\cdots+x_{k_n}} = f(x_1+\cdots+x_{k_n}) = f(x_{k_n}) < f(x_1)+\cdots+f(x_{k_n})$$

because $x_{k_n} < x_1 + x_2 + \cdots + x_{k_n} < x_{k_{n+1}}$,for sufficiently large n. This is contradiction. □

Theorem 2. *Let $f \in R_\infty$ and $\varepsilon > 0$ then*

$$f(1)+f(2)+\cdots+f(n) \ll f(n^{1+\varepsilon}), (n \to \infty).$$

Proof. Let $k > \frac{1}{\varepsilon}$ and $g(x) = f(\sqrt[k]{x})$ then $g \in R_\infty$ and using Theorem 1 we have

$$g(1)+g(2^k)+\cdots+g(n^k) \ll g(1+2^k+\cdots+n^k) = g\left(n^{k+1}O(1)\right), \qquad (n \to \infty).$$

Finally using (2)

$$\begin{aligned} f(1)+f(2)+\cdots+f(n) &\ll g\left(n^{k+1}O(1)\right) = f(n^{1+\frac{1}{k}}O(1)) \\ &\ll f(n^{1+\varepsilon}), \qquad (n \to \infty). \end{aligned}$$ □

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